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An $SL(2,\mathbb{R})$ Model of Constrained Systems: Algebraic Constraint Quantization

Michael Trunk

Universität Freiburg
Fakultät für Physik
Hermann–Herder–Str. 3
D–79104 Freiburg
Germany

e-mail: trunk@physik.uni-freiburg.de

Abstract

A reparametrization invariant model, introduced recently by Montesinos, Rovelli and Thiemann, possessing an $SL(2,\mathbb{R})$ gauge symmetry is treated along the guidelines of an algebraic constraint quantization scheme that translates the vanishing of the constraints into representation conditions for the algebra of observables. The application of this algebraic scheme to the $SL(2,\mathbb{R})$ model yields an unambiguous identification of the physical representation of the algebra of observables.

1 Introduction

In Ref. [MRT], Montesinos, Rovelli, and Thiemann have introduced a reparametrization invariant model with an $SL(2,\mathbb{R})$ gauge symmetry which, in a finite-dimensional context, "mimicks the constraint structure" of general relativity.

In this note I do not want to enter into a discussion of the physical significance of the model. Rather, I will regard the system from the point of view of an algebraic scheme for the quantization of constrained systems (presented in Ref. [Tr]), which lays emphasis on the quantization of observable quantities, translating the "vanishing" of the constraints into representation conditions for the algebra of observables. The $SL(2,\mathbb{R})$ model presents an interesting example for the application of the algebraic scheme, because the model displays a peculiar structural feature that sheds some light upon the working of the method (for a retrospective characterization of this feature, *cf* the conclusions). Also, the treatment of the model presents an application of the algebraic constraint quantization scheme to a reparametrization invariant system with a non-trivial gauge group (as opposed to the – almost obvious – application to the free relativistic particle), possessing constraints which are quadratic in the momenta.

The construction of the quantum theory of the $SL(2,\mathbb{R})$ model according to the algebraic constraint quantization scheme results in an unambiguous identification of the physical representation of the algebra of observables. Moreover, I show how the construction of the quantum theory according to the Dirac quantization scheme, that has been left unfinished in Ref. [MRT], can be completed: In contrast to the approach followed here, the authors of Ref. [MRT] concentrate on the "quantization" of the constraints (*i.e.*, of unobservable quantities) which does not automatically yield the correct quantization of the observables.

2 Classical analysis of the model

2.1 Constraints and observables

The configuration space of the model presented in Ref. [MRT] is $\mathbb{R}^4$, parametrized by the Euclidean variables $\vec{u} = (u_1, u_2)^T$ and $\vec{v} = (v_1, v_2)^T$. The model is defined by the Lagrangian

$$L = \sqrt{\vec{v}^2(\dot{\vec{u}} - \lambda\vec{u})^2} + \sqrt{\vec{u}^2(\dot{\vec{v}} + \lambda\vec{v})^2}.$$

The "multiplier" λ can be eliminated with the help of its equation of motion (two of the multipliers of Ref. [MRT] have already been eliminated), *i.e.* by solving the equation $\partial L/\partial\lambda = 0$ for λ: $\lambda = \tilde{\lambda}(\vec{u}, \dot{\vec{u}}, \vec{v}, \dot{\vec{v}})$. The function $\tilde{\lambda}$ being homogeneous of order one in the velocities, $\tilde{\lambda}(\vec{u}, \alpha\dot{\vec{u}}, \vec{v}, \alpha\dot{\vec{v}}) = \alpha\,\tilde{\lambda}(\vec{u}, \dot{\vec{u}}, \vec{v}, \dot{\vec{v}})$, the action $S = \int \mathrm{d}t\, L$ is invariant under reparametrizations of t.

As a consequence of the definition of the canonical momenta,

$$\vec{p} := \frac{\partial L}{\partial \dot{\vec{u}}} = \left.\frac{\vec{v}^2\,(\dot{\vec{u}} - \lambda\vec{u})}{\sqrt{\vec{v}^2(\dot{\vec{u}} - \lambda\vec{u})^2}}\right|_{\lambda = \tilde{\lambda}(\vec{u}, \dot{\vec{u}}, \vec{v}, \dot{\vec{v}})}, \qquad \vec{\pi} := \frac{\partial L}{\partial \dot{\vec{v}}} = \left.\frac{\vec{u}^2\,(\dot{\vec{v}} + \lambda\vec{v})}{\sqrt{\vec{v}^2(\dot{\vec{v}} + \lambda\vec{v})^2}}\right|_{\lambda = \tilde{\lambda}(\vec{u}, \dot{\vec{u}}, \vec{v}, \dot{\vec{v}})},$$

there are three primary constraints

$$\begin{aligned}
\mathsf{L}_1 &= \frac{1}{2}\,(\mathsf{H}_1 + \mathsf{H}_2) = \frac{1}{4}\,(\vec{p}^2 - \vec{u}^2 + \vec{\pi}^2 - \vec{v}^2) = 0,\\
\mathsf{L}_2 &= \frac{1}{2}\,\mathsf{D} = \frac{1}{2}\,(\vec{u}\cdot\vec{p} - \vec{v}\cdot\vec{\pi}) = 0,\\
\mathsf{L}_3 &= \frac{1}{2}\,(\mathsf{H}_1 - \mathsf{H}_2) = \frac{1}{4}\,(\vec{p}^2 + \vec{u}^2 - \vec{\pi}^2 - \vec{v}^2) = 0.
\end{aligned}$$

The canonical Hamiltonian of the model vanishes, $H = \vec{p}\cdot\dot{\vec{u}}+\vec{\pi}\cdot\dot{\vec{v}}-L = 0$. The constraints are first–class, their Poisson commutation relations close to form a realization of the Lie algebra $sl(2,\mathbb{R}) = so(2,1) = su(1,1)$:

$$\{\mathsf{L}_a, \mathsf{L}_b\} = \varepsilon_{ab}{}^c\,\mathsf{L}_c, \qquad\qquad \varepsilon_{ab}{}^c = g^{cd}\varepsilon_{abd}, \qquad \varepsilon_{123} = 1, \qquad a, b, c, d \in \{1, 2, 3\},$$

where g^{ab} is the inverse of the $so(2,1)$–metric $g_{ab} = \mathrm{diag}(+,+,-)$, and repeated indices are summed over.

The infinitesimal canonical transformations generated by the constraints can be integrated to yield an action of the gauge group $SL(2,\mathbb{R})$ on the space $\mathbb{R}^8 = T^*\mathbb{R}^4$: it acts in its defining two-dimensional representation on the vectors

$$\vec{x}^{(1)} = \begin{pmatrix} u_1 \\ p_1 \end{pmatrix}, \qquad \vec{x}^{(2)} = \begin{pmatrix} u_2 \\ p_2 \end{pmatrix}, \qquad \vec{x}^{(3)} = \begin{pmatrix} \pi_1 \\ v_1 \end{pmatrix}, \qquad \vec{x}^{(4)} = \begin{pmatrix} \pi_2 \\ v_2 \end{pmatrix},$$

i.e., $(\vec{x}^{(1)}, \ldots, \vec{x}^{(4)}) \stackrel{g\in SL(2,\mathbb{R})}{\longrightarrow} (g\vec{x}^{(1)}, \ldots, g\vec{x}^{(4)})$.

2.2 Regularization

The map $\mathsf{L} : \mathbb{R}^8 \to so^*(2,1)$, $(\vec{u}, \vec{v}, \vec{p}, \vec{\pi}) \mapsto \mathsf{L}_a(\vec{u}, \vec{v}, \vec{p}, \vec{\pi})$ ($so^*(2,1)$ is the dual of the Lie algebra $so(2,1)$; L is the so-called momentum map), is not free of critical points where the constraints L_a cease to be functionally independent. A point $q \in \mathbb{R}^8$ is a critical point of L if and only if the two vectors

$$\xi^{(1)} = (u_1, u_2, \pi_1, \pi_2)^T, \qquad\qquad \xi^{(2)} = (p_1, p_2, v_1, v_2)^T,$$

are linearly dependent. In order to avoid the occurrence of "physical states" which cannot be interpreted in terms of observables (see below) the set $\mathcal{S}$ of critical points of L, that is stable under the action of $SL(2,\mathbb{R})$, must be removed from $\mathbb{R}^8$. Consequently, the phase space of the system is $\mathcal{P} = (\mathbb{R}^8 \setminus \mathcal{S}, \bar{\omega})$, with the symplectic form $\bar{\omega}$ that is induced from the canonical symplectic form ω on $T^*\mathbb{R}^4$.

2.3 Observables

The action of $SL(2,\mathbb{R})$ leaves invariant the determinants $\mathsf{O}_{ij} := \det(\vec{x}^{(i)}, \vec{x}^{(j)}) = -\mathsf{O}_{ji}$. Therefore, the (continuous) fundamental observables [Tr] of the system can be chosen to be

$$\mathsf{M}_1 := \mathsf{O}_{23} = u_2 v_1 - p_2\pi_1, \quad \mathsf{M}_2 := \mathsf{O}_{31} = -u_1 v_1 + p_1\pi_1, \quad \mathsf{M}_3 := \mathsf{O}_{12} = u_1 p_2 - u_2 p_1,$$

$$\mathsf{N}_1 := \mathsf{O}_{14} = u_1 v_2 - p_1\pi_2, \quad \mathsf{N}_2 := \mathsf{O}_{24} = u_2 v_2 - p_2\pi_2, \quad \mathsf{N}_3 := \mathsf{O}_{43} = v_1\pi_2 - v_2\pi_1.$$

The Poisson commutation relations of the fundamental observables close to form a realization of the Lie algebra $so(2,2)$:

$$\{\mathsf{M}_a, \mathsf{M}_b\} = \varepsilon_{ab}{}^c\,\mathsf{M}_c, \qquad \{\mathsf{M}_a, \mathsf{N}_b\} = \varepsilon_{ab}{}^c\,\mathsf{N}_c, \qquad \{\mathsf{N}_a, \mathsf{N}_b\} = \varepsilon_{ab}{}^c\,\mathsf{M}_c,$$

which is isomorphic to $so(2,1) \times so(2,1)$, as can be seen by defining $\mathsf{Q}_a := \frac{1}{2}(\mathsf{M}_a + \mathsf{N}_a)$, $\mathsf{P}_a := \frac{1}{2}(\mathsf{M}_a - \mathsf{N}_a)$, whence the commutation relations read

$$\{\mathsf{Q}_a, \mathsf{Q}_b\} = \varepsilon_{ab}{}^c\,\mathsf{Q}_c, \qquad \{\mathsf{Q}_a, \mathsf{P}_b\} = 0, \qquad \{\mathsf{P}_a, \mathsf{P}_b\} = \varepsilon_{ab}{}^c\,\mathsf{P}_c.$$

Alternatively, the commutation relations can be written in the standard form

$$\{\mathsf{Q}_+, \mathsf{Q}_-\} = i\mathsf{Q}_3, \qquad \{\mathsf{Q}_3, \mathsf{Q}_\pm\} = \mp i\mathsf{Q}_\pm, \qquad \mathsf{Q}_\pm := \frac{1}{\sqrt{2}}(\mathsf{Q}_1 \pm i\mathsf{Q}_2), \tag{1}$$

and likewise for P. Again, the infinitesimal canonical transformations generated by the fundamental observables can be integrated, this time to yield an action of the group $SO_0(2,2)$ (the connected component of $SO(2,2)$ containing the identity) on the phase space $\mathcal{P}$, acting in its defining four-dimensional representation on the vectors $\xi^{(1)}$ and $\xi^{(2)}$:

$$(\xi^{(1)}, \xi^{(2)}) \overset{h \in SO_0(2,2)}{\longrightarrow} (h\xi^{(1)}, h\xi^{(2)})$$

(obviously, the set $\mathcal{S}$ is stable under the action of $SO_0(2,2)$ on $\mathbb{R}^8$). In addition, there are the discrete fundamental "observables" (reflections) R_{xy}, $(x,y) \in \{(u_1,p_1),(u_2,p_2),(\pi_1,v_1),(\pi_2,v_2)\}$, which change the sign of the coordinates x and y, leaving all other coordinates invariant. Only two of them are independent (taking into account the action of $SO_0(2,2)$), say $\mathsf{R}_1 := \mathsf{R}_{\pi_2 v_2}$ and $\mathsf{R}_2 := \mathsf{R}_{\pi_2 v_2} \circ \mathsf{R}_{u_1 p_1}$. Together with Q_a and P_a they generate an action of the group $O(2,2)$ on the phase space. The action of the discrete observables R_1 and R_2 on the continuous observables is given by

$$\mathsf{R}_1 :\ \mathsf{Q}_a \longleftrightarrow \mathsf{P}_a, \qquad \mathsf{R}_2 :\ (\mathsf{Q}_3,\mathsf{P}_3) \longleftrightarrow (-\mathsf{Q}_3,-\mathsf{P}_3), \quad (\mathsf{Q}_\pm,\mathsf{P}_\pm) \longleftrightarrow (\mathsf{Q}_\mp,\mathsf{P}_\mp). \tag{2}$$

Finally, there is a $*$–involution (complex conjugation) on the classical algebra of observables (*i.e.*, the Poisson algebra generated polynomially by the fundamental observables Q_a and P_a that comprises the action of R_1 and R_2), acting on the fundamental observables like

$$(\mathsf{Q}_3,\mathsf{Q}_\pm,\mathsf{P}_3,\mathsf{P}_\pm,\mathsf{R}_1,\mathsf{R}_2) \longrightarrow (\mathsf{Q}_3^*,\mathsf{Q}_\pm^*,\mathsf{P}_3^*,\mathsf{P}_\pm^*,\mathsf{R}_1^*,\mathsf{R}_2^*) = (\mathsf{Q}_3,\mathsf{Q}_\mp,\mathsf{P}_3,\mathsf{P}_\mp,\mathsf{R}_1,\mathsf{R}_2). \tag{3}$$

2.4 The observable content of the constraints

By definition [Tr], the observable content of the constraints comprises the conditions which are imposed on the (Casimir) invariants of the algebra of observables by the "vanishing" of the constraints (via functional dependencies between the invariants of the algebra of observables and the generalized Casimir elements of the constraints [Tr]). Upon quantization, these conditions are turned into representation conditions which select the physical representation(s) of the algebra of observables.

The Lie algebra $so(2,2)$ possesses two quadratic Casimir "operators", $\mathsf{Q}^2 := g^{ab}\mathsf{Q}_a\mathsf{Q}_b = \mathsf{Q}_+\mathsf{Q}_- + \mathsf{Q}_-\mathsf{Q}_+ - \mathsf{Q}_3^2$ and $\mathsf{P}^2 := g^{ab}\mathsf{P}_a\mathsf{P}_b$. If $\mathsf{Q}^2 \le 0$ ($\mathsf{P}^2 \le 0$), the sign of Q_3 (P_3) is another invariant (where the sign can be equal to either plus one, zero, or minus one).

In the realization of the Lie algebra $so(2,2)$ by the observables of the model at hand, there are two identities which reflect the functional dependencies between the Casimirs of the algebra of observables and the Casimir $\mathsf{L}^2 := g^{ab}\mathsf{L}_a\mathsf{L}_b$ of the constraint algebra:

$$\mathsf{Q}^2 - \mathsf{L}^2 \equiv 0, \qquad \mathsf{P}^2 - \mathsf{L}^2 \equiv 0. \tag{4}$$

Upon the vanishing of the constraints they induce the identities

$$\mathsf{Q}^2 = 0, \qquad \mathsf{P}^2 = 0. \tag{5}$$

In addition, there are further identities which determine the signs of Q_3 and P_3. These identities involve the generators $\mathsf{Q}_a\mathsf{P}_b$ of a Poisson ideal in the algebra of observables. Only one of them is independent:

$$2\,\mathsf{Q}_3\mathsf{P}_3 - [(\vec{u}^2 - \vec{v}^2)\,\mathsf{L}_1 - (\vec{u}\cdot\vec{p} + \vec{v}\cdot\vec{\pi})\,\mathsf{L}_2 + (\vec{u}^2 + \vec{v}^2)\,\mathsf{L}_3] \equiv 0, \tag{6}$$

in the sense that the others can be obtained from it via (repeated) Poisson bracket operation with Q_a and P_a. The vanishing of the constraints induces the identities

$$\mathsf{Q}_a \mathsf{P}_b = 0, \tag{7}$$

which possess the solutions ($\vee$ is the logical "or")

$$[\,\mathsf{Q}_a = 0 \quad \forall a\,] \;\vee\; [\,\mathsf{P}_a = 0 \quad \forall a\,].$$

These identities can be translated into conditions involving the signs of Q_3 and P_3:

$$[\,\mathrm{sign}(\mathsf{Q}_3) = 0\,] \;\vee\; [\,\mathrm{sign}(\mathsf{P}_3) = 0\,]. \tag{8}$$

Now, the observables $\mathsf{O}_{ij} = \mathsf{O}_{ij}(\vec{u}, \vec{v}, \vec{p}, \vec{\pi})$, regarded as functions on $\mathbb{R}^8$, vanish for all pairs (ij) of the indices if and only if $(\vec{u}, \vec{v}, \vec{p}, \vec{\pi}) \in \mathcal{S}$. Therefore, on the phase space $\mathcal{P}$ the conditions (8) take on the form

$$(\,[\,\mathrm{sign}(\mathsf{Q}_3) = 0\,] \;\vee\; [\,\mathrm{sign}(\mathsf{P}_3) = 0\,]\,) \;\neg\; (\,[\,\mathrm{sign}(\mathsf{Q}_3) = 0\,] \;\wedge\; [\,\mathrm{sign}(\mathsf{P}_3) = 0\,]\,) \tag{9}$$

($\wedge$ is "and", $\neg$ is "not"). The latter conditions, together with the induced identities (5), express the observable content of the constraints.

2.5 The structure of the space of physical states

The topological structure of the space of physical states (*i.e.*, the reduced phase space) is determined entirely by the observable content of the constraints. As the action of the group $SO_0(2,2)$ on (the linear span of) the observables Q_a and P_a coincides with the coadjoint action of $SO_0(2,2)$ on the dual $so^*(2,2) = so^*(2,1)_Q \times so^*(2,1)_P$ of its Lie algebra $so(2,2) = so(2,1)_Q \times so(2,1)_P$, the space of physical states is the disjoint union of several coadjoint orbits for $SO_0(2,2)$ which, in their turn, are labelled by the values of the invariants.

The identity $\mathsf{Q}^2 = 0$ characterizes the "light cone" $\mathcal{C}_Q$ in $so^*(2,1)_Q$, which falls into three pieces: the "forward light cone" $\mathcal{C}_Q^+$ ($\mathrm{sign}(\mathsf{Q}_3) = +1$), the "backward light cone" $\mathcal{C}_Q^-$ ($\mathrm{sign}(\mathsf{Q}_3) = -1$), and the origin $\mathcal{C}_Q^0$ ($\mathrm{sign}(\mathsf{Q}_3) = 0$) (and similarly for P). Therefore, according to the conditions (5) and (9), the space of physical states consists of four pieces (orbits for $SO_0(2,2)$):

$$\mathcal{C}_I = \mathcal{C}_Q^+ \times \mathcal{C}_P^0, \qquad \mathcal{C}_{II} = \mathcal{C}_Q^- \times \mathcal{C}_P^0, \qquad \mathcal{C}_{III} = \mathcal{C}_Q^0 \times \mathcal{C}_P^+, \qquad \mathcal{C}_{IV} = \mathcal{C}_Q^0 \times \mathcal{C}_P^-.$$

These four pieces constitute one orbit $\mathcal{C} = \mathcal{C}_I \cup \mathcal{C}_{II} \cup \mathcal{C}_{III} \cup \mathcal{C}_{IV}$ for $O(2,2)$, consisting of four connected components that are mapped onto one another by the discrete observables R_1, R_2 and $\mathsf{R}_1 \circ \mathsf{R}_2$ (corresponding to the four connected components of the group $O(2,2)$).

3 Construction of the quantum theory

I will now construct the quantum theory of the model, following the quantization scheme outlined in Ref. [Tr].

3.1 The quantum algebra of observables

The first step is the construction of the quantum algebra of observables. The quantum algebra of observables is generated polynomially by the fundamental observables $\{\hat{\mathsf{Q}}_a, \hat{\mathsf{P}}_a\}$, the unambiguous quantum analogs of the classical fundamental observables $\{\mathsf{Q}_a, \mathsf{P}_a\}$, and an action of the discrete observables $\{\hat{\mathsf{R}}_1, \hat{\mathsf{R}}_2\}$, the quantum analogs of the classical observables $\{\mathsf{R}_1, \mathsf{R}_2\}$, has to be defined on it. Up to quantum corrections, the commutation relations of the observables $\hat{\mathsf{Q}}_a$ and $\hat{\mathsf{P}}_a$ have to be inferred from the Poisson commutation relations of the corresponding classical observables.

The only quantum correction of the classical $so(2,2)$ commutation relations compatible with the principles formulated in Ref. [Tr] would be a central extension of the algebra $so(2,2)$. However, as the second cohomology of the algebra $so(2,2)$ is trivial ($so(2,2)$ being semi–simple), there is no non–trivial central extension available (*cf* [Wo]), and the commutation relations have to be taken to be

$$[\hat{\mathsf{Q}}_a, \hat{\mathsf{Q}}_b] = i\hbar\, \varepsilon_{ab}{}^c\, \hat{\mathsf{Q}}_c, \qquad [\hat{\mathsf{Q}}_a, \hat{\mathsf{P}}_b] = 0, \qquad [\hat{\mathsf{P}}_a, \hat{\mathsf{P}}_b] = i\hbar\, \varepsilon_{ab}{}^c\, \hat{\mathsf{P}}_c.$$

The action of the discrete observables cannot pick up quantum corrections. According to equation (2), it is given by ($\hat{\mathsf{Q}}_\pm$ and $\hat{\mathsf{P}}_\pm$ are defined as in the classical theory, *cf* (1))

$$\hat{\mathsf{R}}_1\,(\hat{\mathsf{Q}}_a, \hat{\mathsf{P}}_a)\,\hat{\mathsf{R}}_1 = (\hat{\mathsf{P}}_a, \hat{\mathsf{Q}}_a), \quad \hat{\mathsf{R}}_2\,(\hat{\mathsf{Q}}_3, \hat{\mathsf{P}}_3)\,\hat{\mathsf{R}}_2 = (-\hat{\mathsf{Q}}_3, -\hat{\mathsf{P}}_3), \quad \hat{\mathsf{R}}_2\,(\hat{\mathsf{Q}}_\pm, \hat{\mathsf{P}}_\pm)\,\hat{\mathsf{R}}_2 = (\hat{\mathsf{Q}}_\mp, \hat{\mathsf{P}}_\mp). \quad (10)$$

3.2 The observable content of the constraints

The next step is the determination of the form that the conditions (5) and (9) assume upon quantization. The Casimir operators of $so(2,2)$ being unambiguously defined, $\hat{\mathsf{Q}}^2 := g^{ab}\hat{\mathsf{Q}}_a\hat{\mathsf{Q}}_b = \hat{\mathsf{Q}}_+\hat{\mathsf{Q}}_- + \hat{\mathsf{Q}}_-\hat{\mathsf{Q}}_+ - \hat{\mathsf{Q}}_3^2$, $\hat{\mathsf{P}}^2 := g^{ab}\hat{\mathsf{P}}_a\hat{\mathsf{P}}_b$, the only consistent quantum corrections of the identities (5) are by constants:

$$\hat{\mathsf{Q}}^2 - c_Q\hbar^2 = 0, \qquad \hat{\mathsf{P}}^2 - c_P\hbar^2 = 0, \qquad c_Q = \text{const.},\ c_P = \text{const.}.$$

However, these corrections have to vanish as can be seen by the following argument: The conditions (9), concerning the signs of Q_3 and P_3, cannot acquire quantum corrections, *i.e.* their quantum counterparts have to read

$$(\,[\,\mathrm{sign}(\hat{\mathsf{Q}}_3) = 0\,]\ \vee\ [\,\mathrm{sign}(\hat{\mathsf{P}}_3) = 0\,]\,)\ \neg\ (\,[\,\mathrm{sign}(\hat{\mathsf{Q}}_3) = 0\,]\ \wedge\ [\,\mathrm{sign}(\hat{\mathsf{P}}_3) = 0\,]\,) \qquad (11)$$

($\mathrm{sign}(\hat{\mathsf{O}}) = -1, 0, +1$ means that the operator $\hat{\mathsf{O}}$ is negative definite, zero, or positive definite, respectively). But the condition $\mathrm{sign}(\hat{\mathsf{Q}}_3) = 0$ is only compatible with the identity $\hat{\mathsf{Q}}^2 = 0$, and analogously for $\hat{\mathsf{P}}^2$, thereby enforcing the identities

$$\hat{\mathsf{Q}}^2 = 0, \qquad \hat{\mathsf{P}}^2 = 0. \qquad (12)$$

3.3 The physical representations of the algebra of observables

The final step is the identification of the physical representations of the algebra of observables, making use of the observable content of the constraints as it is expressed by the conditions (11) and (12), and of the hermiticity relations

$$(\hat{\mathsf{Q}}_3, \hat{\mathsf{Q}}_\pm, \hat{\mathsf{P}}_3, \hat{\mathsf{P}}_\pm, \hat{\mathsf{R}}_1, \hat{\mathsf{R}}_2) \longrightarrow (\hat{\mathsf{Q}}_3^\dagger, \hat{\mathsf{Q}}_\pm^\dagger, \hat{\mathsf{P}}_3^\dagger, \hat{\mathsf{P}}_\pm^\dagger, \hat{\mathsf{R}}_1^\dagger, \hat{\mathsf{R}}_2^\dagger) = (\hat{\mathsf{Q}}_3, \hat{\mathsf{Q}}_\mp, \hat{\mathsf{P}}_3, \hat{\mathsf{P}}_\mp, \hat{\mathsf{R}}_1, \hat{\mathsf{R}}_2),$$

which implement the classical $*$–relations (3) into the quantum theory and require the representations to be hermitian (corresponding to unitary representations of the group $O(2,2)$). As the hermitian representations of the Lie algebra $so(2,2)$ are the tensor products of the hermitian representations of the factors $so(2,1)_Q$ and $so(2,1)_P$, it is sufficient to determine the latter.

There are three hermitian irreducible representations of $so(2,1)_Q$ which are selected uniquely by the conditions $\hat{\mathsf{Q}}^2 = 0$ and $\mathrm{sign}(\hat{\mathsf{Q}}_3) = -1,\ 0,\ +1$ (for a classification of the hermitian irreducible representations of $so(2,1)$ see Ref. [Wy]): the representation $D_Q^-(-1)$, the trivial representation $\mathbb{1}_Q$, and the representation $D_Q^+(-1)$, respectively (in the notation of Ref. [Wy]). The representations are spanned by orthonormal states ($m \in \mathbb{N}$)

$$|-m\rangle_Q \quad \text{for} \quad D_Q^-(-1), \qquad |0\rangle_Q \quad \text{for} \quad \mathbb{1}_Q, \qquad |m\rangle_Q \quad \text{for} \quad D_Q^+(-1).$$

The action of $\hat{\mathsf{Q}}_3$ and $\hat{\mathsf{Q}}_\pm$ on these states is given by ($n = -m, 0, m$)

$$\hat{\mathsf{Q}}_3\,|n\rangle_Q = \hbar n\,|n\rangle_Q, \qquad \hat{\mathsf{Q}}_\pm\,|n\rangle_Q = \frac{\hbar}{\sqrt{2}}\,\sqrt{n(n\pm 1)}\,|n\pm 1\rangle_Q. \tag{13}$$

Analogously, there are three hermitian irreducible representations $D_P^-(-1)$, $\mathbb{1}_P$, $D_P^+(-1)$ of $so(2,1)_P$, selected uniquely by the conditions $\hat{\mathsf{P}}^2 = 0$ and $\mathrm{sign}(\hat{\mathsf{P}}_3) = -1,\ 0,\ +1$, respectively.

As a consequence, the observable content of the constraints, conditions (11) and (12), selects uniquely four hermitian irreducible representations of the Lie algebra $so(2,2)$, corresponding to the four orbits which make up the classical space of physical states, namely $D_I = D_Q^+(-1) \otimes \mathbb{1}_P$, $D_{II} = D_Q^-(-1) \otimes \mathbb{1}_P$, $D_{III} = \mathbb{1}_Q \otimes D_P^+(-1)$, $D_{IV} = \mathbb{1}_Q \otimes D_P^-(-1)$, spanned by the states $|m,0\rangle = |m\rangle_Q \otimes |0\rangle_P$, $|-m,0\rangle = |-m\rangle_Q \otimes |0\rangle_P$, $|0,m\rangle = |0\rangle_Q \otimes |m\rangle_P$, $|0,-m\rangle = |0\rangle_Q \otimes |-m\rangle_P$, respectively. The physical Hilbert space $\mathcal{H}_{phys}$ is the direct sum of (the carrier spaces of) these four representations.

Finally, defining the action of the discrete observables $\hat{\mathsf{R}}_1$ and $\hat{\mathsf{R}}_2$ on $\mathcal{H}_{phys}$ by

$$\hat{\mathsf{R}}_1\,|n,0\rangle = |0,n\rangle, \qquad \hat{\mathsf{R}}_1\,|0,n\rangle = |n,0\rangle, \qquad \hat{\mathsf{R}}_2\,|n,0\rangle = |-n,0\rangle, \qquad \hat{\mathsf{R}}_2\,|0,n\rangle = |0,-n\rangle,$$

the requirements (10) are satisfied, and $\mathcal{H}_{phys}$ carries one unitary irreducible representation $D = D_I \oplus D_{II} \oplus D_{III} \oplus D_{IV}$ of the group $O(2,2)$.

4 Comparison with the Dirac quantization

The phase space $\mathcal{P}$ not being a cotangent bundle, the canonical Dirac quantization scheme does not seem to be an appropriate setting for the construction of the quantum theory of the $SL(2,\mathbb{R})$ model. Nevertheless, it is possible to - at least abstractly - complete the Dirac quantization that has been left unfinished in Ref. [MRT].

4.1 Constraints and physical states

The authors of Ref. [MRT] represent the constraints as symmetric operators $\hat{\mathsf{H}}_1$ ($=: \hat{\mathsf{L}}_1 + \hat{\mathsf{L}}_3$), $\hat{\mathsf{H}}_2$ ($=: \hat{\mathsf{L}}_1 - \hat{\mathsf{L}}_3$) and $\hat{\mathsf{D}}$ ($=: 2\,\hat{\mathsf{L}}_2$) with the $so(2,1)$ commutation relations $[\hat{\mathsf{L}}_a, \hat{\mathsf{L}}_b] = i\hbar\,\varepsilon_{ab}{}^c\,\hat{\mathsf{L}}_c$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^4, \mathrm{d}^2u\,\mathrm{d}^2v)$. The concrete expressions for these operators can be obtained from the classical ones upon substituting the multiplicative operators $\hat{\vec{u}} = \vec{u}$ and $\hat{\vec{v}} = \vec{v}$ for the classical coordinates $\vec{u}$ and $\vec{v}$, and the differential operators $\hat{\vec{p}} = -i\hbar\,\vec{\nabla}_u$ and $\hat{\vec{\pi}} = -i\hbar\,\vec{\nabla}_v$ for the classical momenta $\vec{p}$ and $\vec{\pi}$, leaving the order of the coordinates and momenta unchanged. They determine the "physical states" of the system as distributional solutions of the differential equations $\hat{\mathsf{L}}_a\,\Psi(\vec{u},\vec{v}) = 0$. In polar coordinates, $\vec{u} = (u\,\cos\alpha, u\,\sin\alpha)^T$, $\vec{v} = (v\,\cos\beta, v\,\sin\beta)^T$, these solutions are given by

$$\Psi_{m,\epsilon}(u,v,\alpha,\beta) = e^{im(\alpha+\epsilon\beta)}\,J_m(uv/\hbar),$$

where $\epsilon = \pm 1$, $m \in \mathbb{Z}$, and $J_m(z)$ are Bessel functions. (By the way: there are also other "physical states", *e.g.* the states $\Psi_\pm = \exp\{\pm i\,\vec{u}\cdot\vec{v}/\hbar\}$ and all states that can be obtained from them via the action of observables.) The authors do not specify an inner product on these states, *i.e.* they do not construct a physical Hilbert space. Furthermore, they represent only the generators of the maximal compact subgroup $SO(2)\times SO(2)$ of $SO_0(2,2)$ as symmetric operators on $\mathcal{H}$, which is not sufficient for the determination of the representations of the full algebra of observables.

4.2 Observables

The classical observables Q_a and P_a can be represented as symmetric operators $\hat{\mathsf{Q}}_a$ and $\hat{\mathsf{P}}_a$ on $\mathcal{H}$ in the same way as the constraints (*i.e.*, by the same substitutions; there are no ordering ambiguities). The observables obey the commutation relations

$$[\hat{\mathsf{Q}}_a,\hat{\mathsf{Q}}_b] = i\hbar\,\varepsilon_{ab}{}^{c}\,\hat{\mathsf{Q}}_c, \qquad [\hat{\mathsf{Q}}_a,\hat{\mathsf{P}}_b] = 0, \qquad [\hat{\mathsf{P}}_a,\hat{\mathsf{P}}_b] = i\hbar\,\varepsilon_{ab}{}^{c}\,\hat{\mathsf{P}}_c$$

and commute with the constraint operators. With the concrete expressions for the observables and constraints the quantum counterparts of the classical identities (4) and (6) can be computed explicitly. They read

$$\hat{\mathsf{Q}}^2 - \hat{\mathsf{L}}^2 \equiv 0, \qquad\qquad \hat{\mathsf{P}}^2 - \hat{\mathsf{L}}^2 \equiv 0 \tag{14}$$

and

$$2\,\hat{\mathsf{Q}}_3\hat{\mathsf{P}}_3 - [(\hat{\vec{u}}^{\,2} - \hat{\vec{v}}^{\,2})\,\hat{\mathsf{L}}_1 - (\hat{\vec{u}}\cdot\hat{\vec{p}} + \hat{\vec{v}}\cdot\hat{\vec{\pi}})\,\hat{\mathsf{L}}_2 + (\hat{\vec{u}}^{\,2} + \hat{\vec{v}}^{\,2})\,\hat{\mathsf{L}}_3] \equiv 0, \tag{15}$$

Again, the missing identities can be obtained from (15) via (repeated) commutator operation with $\hat{\mathsf{Q}}_a$ and $\hat{\mathsf{P}}_a$. On the linear space $\mathcal{V}$ of "physical states" the identities (14) and (15) induce the identities

$$\hat{\mathsf{Q}}^2 = 0, \qquad \hat{\mathsf{P}}^2 = 0, \qquad \hat{\mathsf{Q}}_a\hat{\mathsf{P}}_b = 0,$$

which coincide with the classical identities (5) and (7).

4.3 Representations

The representations of $so(2,1)_Q\times so(2,1)_P$ which are realized on the linear space $\mathcal{V}$ of "physical states" can be identified by explicitly computing the action of $\hat{\mathsf{Q}}_3$, $\hat{\mathsf{Q}}_\pm$, $\hat{\mathsf{P}}_3$ and $\hat{\mathsf{P}}_\pm$ on the states $\Psi_{m,\epsilon}$. In polar coordinates these operators are given by ($\hat{\mathsf{O}}_a(+1) := \hat{\mathsf{Q}}_a$, $\hat{\mathsf{O}}_a(-1) := \hat{\mathsf{P}}_a$)

$$\hat{\mathsf{O}}_3(\epsilon) = -\frac{i\hbar}{2}\,(\partial_\alpha + \epsilon\partial_\beta), \qquad \hat{\mathsf{O}}_\pm(\epsilon) = \frac{\hbar}{\sqrt{8}}\,e^{\pm i(\alpha+\epsilon\beta)}\left[\frac{\epsilon}{v}\,\partial_u\partial_\beta + \frac{1}{u}\,\partial_v\partial_\alpha \mp i\left(uv + \partial_u\partial_v - \frac{\epsilon}{uv}\,\partial_\alpha\partial_\beta\right)\right].$$

The action of the operators $\hat{\mathsf{Q}}_a$ on the states $\Psi_{m,-}$ is trivial: $\hat{\mathsf{Q}}_a\,\Psi_{m,-} = 0$. On the states $\Psi_{m,+}$ $\hat{\mathsf{Q}}_3$ and $\hat{\mathsf{Q}}_\pm$ act as

$$\hat{\mathsf{Q}}_3\,\Psi_{m,+} = \hbar m\,\Psi_{m,+}, \qquad \hat{\mathsf{Q}}_\pm\,\Psi_{m,+} = \frac{\mp i}{\sqrt{2}}\,\hbar m\,\Psi_{m\pm 1,+} \tag{16}$$

(and similarly for $\hat{\mathsf{P}}_3$ and $\hat{\mathsf{P}}_\pm$, with the rôles of $\Psi_{m,+}$ and $\Psi_{m,-}$ interchanged). Equation (16) (and its counterpart for $\hat{\mathsf{P}}_\pm$) shows that the linear space $\mathcal{V}$ contains five invariant subspaces for the action of $so(2,2)$: $\mathcal{V}_Q^+ = \{\Psi_{m,+}|m \geq 0\}$, $\mathcal{V}_Q^- = \{\Psi_{m,+}|m \leq 0\}$, $\mathcal{V}_P^+ = \{\Psi_{m,-}|m \geq 0\}$, $\mathcal{V}_P^- = \{\Psi_{m,-}|m \leq 0\}$, and $\mathcal{V}_0 = \{\Psi_{0,+} = \Psi_{0,-}\}$. The space $\mathcal{V}$ is not the direct sum of these invariant subspaces, as all spaces have the subspace $\mathcal{V}_0$ in common. This means, that the representation of $so(2,2)$ on $\mathcal{V}$ is reducible, but not fully reducible (*i.e.*, it is indecomposable).

The invariant subspace $\mathcal{V}_Q^+$ carries a representation of $so(2,1)_Q$. In the notation of Ref. [Wy] it is the representation $D_Q^+(0)$. This representation is also indecomposable, as it contains the invariant subspace $\mathcal{V}_0$ and as the action of $\hat{\mathsf{Q}}_-$ maps $\Psi_{1,+}$ into $\mathcal{V}_0$. This latter fact means that the sign of $\hat{\mathsf{Q}}_3$ is not invariant (it changes from plus one to zero), showing that the observable content of the constraints is not reproduced correctly. Moreover, the representation is not unitary. For, assume that there exists a scalar product $\langle\cdot|\cdot\rangle$ on $\mathcal{V}_Q^+$, such that $\langle\Phi_{m,+}|\Phi_{m',+}\rangle = \delta_{mm'}$ ($m, m' \geq 0$, $\Phi_{m,+} = \mathcal{N}_m\,\Psi_{m,+}$, $\mathcal{N}_m$ non-zero and finite normalization constants) and such that $\hat{\mathsf{Q}}_+^\dagger = \hat{\mathsf{Q}}_-$. Then, from equation (16) one finds

$$\hat{\mathsf{Q}}_-\,\Phi_{1,+} = \frac{i\hbar}{\sqrt{2}}\frac{\mathcal{N}_1}{\mathcal{N}_0}\,\Phi_{0,+}, \qquad \hat{\mathsf{Q}}_+\hat{\mathsf{Q}}_-\,\Phi_{1,+} = 0,$$

which implies the contradiction

$$0 = \langle\Phi_{1,+}|\hat{\mathsf{Q}}_+\hat{\mathsf{Q}}_-|\Phi_{1,+}\rangle = \langle\hat{\mathsf{Q}}_-\,\Phi_{1,+}|\hat{\mathsf{Q}}_-\,\Phi_{1,+}\rangle = \frac{\hbar^2}{2}\left|\frac{\mathcal{N}_1}{\mathcal{N}_0}\right|^2 \neq 0.$$

However, it is possible to (abstractly) introduce a degenerate scalar product on (an abstract linear space that is isomorphic to) $\mathcal{V}_Q^+$ which allows to turn $\mathcal{V}_Q^+$ into a Hilbert space that carries the irreducible representation $D_Q^+(-1)$. For this purpose define states $|m)$, $m \in \mathbb{N}_0$, and abstract operators $\hat{\mathsf{Q}}_3$ and $\hat{\mathsf{Q}}_\pm$ which act on these states as in equation (16), *i.e.* as

$$\hat{\mathsf{Q}}_3\,|m) = \hbar m\,|m), \qquad \hat{\mathsf{Q}}_\pm\,|m) = \frac{\mp i}{\sqrt{2}}\,\hbar m\,|m \pm 1).$$

Introduce a degenerate scalar product $(\cdot|\cdot)$ on these states by requiring

$$(0|0) = 0, \qquad (0|m) = 0, \qquad (m|m') = |N_m|^{-2}\,\delta_{mm'}$$

($m, m' \in \mathbb{N}$, N_m are non-zero and finite complex normalization constants). Denote the equivalence classes (*i.e.*, states modulo zero norm states) of $N_m\,|m)$, $m \in \mathbb{N}$, by $|m\rangle$. The induced scalar product on these states is $\langle m|m'\rangle = \delta_{mm'}$, and the action of the operators $\hat{\mathsf{Q}}_3$ and $\hat{\mathsf{Q}}_\pm$ is given by

$$\hat{\mathsf{Q}}_3\,|m\rangle = \hbar m\,|m\rangle, \qquad \hat{\mathsf{Q}}_\pm\,|m\rangle = \frac{\mp i}{\sqrt{2}}\,\hbar m\,\frac{N_m}{N_{m\pm1}}\,|m \pm 1\rangle.$$

The normalization constants N_m can be fixed (up to a phase) by requiring that $\hat{\mathsf{Q}}_+^\dagger = \hat{\mathsf{Q}}_-$. This yields the condition $|N_m/N_{m+1}|^2 = (m+1)/m$, which can be satisfied by putting (fixing also the phase)

$$N_m = \frac{i^{-m}}{\sqrt{m}}.$$

As a consequence, the operators $\hat{\mathsf{Q}}_\pm$ act on the states $|m\rangle$ as

$$\hat{\mathsf{Q}}_\pm\,|m\rangle = \frac{\hbar}{\sqrt{2}}\,\sqrt{m(m \pm 1)}\,|m \pm 1\rangle,$$

and a comparison with equation (13) reveals, that the Hilbert space $\mathcal{H}_Q^+$, which is spanned by the states $|m\rangle$ and equipped with the scalar product $\langle\cdot|\cdot\rangle$, carries the hermitian irreducible representation $D_Q^+(-1)$ of $so(2,1)_Q$ (or the representation $D_Q^+(-1)\otimes\mathbb{1}_P$ of $so(2,1)_Q\times so(2,1)_P$, *i.e.*, the states $|m\rangle$ can be identified with the states $|m,0\rangle$ of Sec. 3.3).

In an analogous fashion, the invariant subspaces $\mathcal{V}_Q^-$, $\mathcal{V}_P^+$ and $\mathcal{V}_P^-$ can be turned into Hilbert spaces $\mathcal{H}_Q^-$, $\mathcal{H}_P^+$ and $\mathcal{H}_P^-$ which carry the irreducible representations $D_Q^-(-1)\otimes\mathbb{1}_P$, $\mathbb{1}_Q\otimes D_P^+(-1)$ and $\mathbb{1}_Q\otimes D_P^-(-1)$, respectively, thereby establishing a one-to-one correspondence between the basis states of the space $\mathcal{V}-\mathcal{V}_0$ and those of the direct sum of the Hilbert spaces $\mathcal{H}_Q^\pm$ and $\mathcal{H}_P^\pm$. In this way the results of Sec. 3.3 can be reproduced.

5 Conclusions

In this note I hope to have demonstrated that the algebraic method is an effective and natural (in the sense of being well adapted to the problem) tool for the construction of the quantum theory of constrained systems.

Let me point out a peculiar structural feature that makes the $SL(2,\mathbb{R})$ model an interesting example for the application of the method, emphasizing the crucial rôle that is played by the correct identification of the observable content of the constraints: For the determination of the observable content of the constraints it is necessary to take into account not only the direct functional dependencies (equations (4)) between the Casimirs of the algebra of observables and those of the constraint algebra, but also dependencies (equation (6) and its Poisson bracket transforms) between the generators of an ideal within the algebra of observables and generalized Casimir elements of the constraints. However, the conditions that are imposed by the vanishing of the constraints via the latter dependencies can be reformulated as restrictions on the values of the invariants of the algebra of observables, *cf* equation (9).

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Postscript: Some time after the completion of the present note the preprint [LR] appeared, where the authors Louko and Rovelli construct the quantum theory of the $SL(2,\mathbb{R})$ model using the quantization schemes of “Algebraic Quantization” and “Refined Algebraic Quantization”, arriving at essentially the same results as in the treatment given here. A comparison of the conceptual and technical advantages and disadvantages of the various schemes is left to the reader. I just want to point out that the approach presented here does not make use of any “input motivated by the structure of the classical constraints” [LR], and that it does not depend on “making successful choices in the ‘early’ steps” which “may require hindsight from the ‘later’ steps” [LR]. To be sure, just like “Algebraic Quantization” and “Refined Algebraic Quantization” the algebraic quantization scheme followed here is not a “*prescription* for quantization” [LR] which can do without any input, but here the input is taken exclusively from the observable sector of the system in question – in the form of correspondence and consistency requirements which are imposed on *observable* quantities.

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